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An Approach to Derive the Equations of Rigid-Body Motion

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Abstract - **In most engineering mechanics textbooks, the equations of general rigid-body motion are usually arrived at in anon-rigorous way, often through motivation using special cases and generalization. The topic of motions in a moving frame is then usually presented separately afterward. In this paper, the equation for velocity of a rigid body is derived by attaching a frame to the rigid body and utilizing the property of a moving frame. The existence of a unique angular velocity vector is then proved and the equations for the general motion of a rigid body follow logically.**

Keywords -**Rigid body, rigid-body motions, moving frame, angular velocity, velocity, acceleration**

I. INTRODUCTION

A rigid body is an idealized body that does not deform regardless of its motion. Strictly speaking, such body does not exist. However, it is an extremely useful concept for analyzing motions of bodies and parts when deformation is not the main concern. Consequently, the topic of rigid-body motion is being taught in all physics and engineering mechanics courses. The equation describing the motion of a rigid body is well known and its general form can be written as

$$\boldsymbol{x}(t) = \boldsymbol{Q}(t)\boldsymbol{X} + \boldsymbol{c}(t), \qquad (1.1)$$

where $\boldsymbol{x}(t)$ is the position vector of a point in the rigid body at time t, $\boldsymbol{X}$ the position vector of the point in some reference configuration, $\boldsymbol{Q}(t)$ a proper orthogonal tensor, and $\boldsymbol{c}(t)$ a vector function of time. This result can be proved in many ways (see for example [1]). However, the proof is usually too complicated for most first- or second-year students who have to study engineering mechanics. From (1.1), it follows immediately that the velocity and acceleration of any point of the rigid-body have the form

$$\boldsymbol{v} = \boldsymbol{v}_A + \boldsymbol{\omega}\ \boldsymbol{x}\ \boldsymbol{r}_{/A} \qquad (1.2)$$

$$\boldsymbol{a} = \boldsymbol{a}_A + \dot{\boldsymbol{\omega}}\boldsymbol{x}\ \boldsymbol{r}_{/A} + \boldsymbol{\omega}\ (\boldsymbol{\omega} \times \boldsymbol{r}_{/A}) \qquad (1.3)$$

where $\boldsymbol{v}$ and $\boldsymbol{a}$ are the velocity and acceleration vectors, respectively, of the point, $\boldsymbol{v}_A$ and $\boldsymbol{a}_A$ are the velocity and acceleration vectors, respectively, of a reference point A of the body, $\boldsymbol{r}_{/A}$ the relative position vector of the point of interest with respect to A, $\boldsymbol{\omega}$ the angular velocity vector of the motion, and $\dot{\boldsymbol{\omega}}$ the angular acceleration vector. Equation (1.2) and (1.3) are given in most engineering mechanics textbooks (e.g. [2], [3]) and were derived by a different approach in [4].

In most engineering mechanics textbooks, the equations (1.2) and (1.3) are not rigorously derived and are usually arrived at in a round-about way. Firstly, the rotation of a rigid body about a fixed axis is motivated and presented. This is followed by a consideration of two-dimensional motions about a fixed point. The two are then combined to give a general two-dimensional motion of a rigid body in the form (1.2) and (1.3). An extension to the case of three-dimensional rigid-body motion is then assumed. Within the same chapter of rigid-body motion, the topic of motions in a moving frame is usually presented afterward.

In this paper, a different approach is taken to derive the equations of rigid-body motion (1.2) and (1.3) rigorously while keeping in mind the purpose of using it in the instruction of an undergraduate engineering mechanics course.

II. MOTIONS IN A MOVING FRAME

Consider a rotation of a vector $\boldsymbol{p}$ around an axis. It can easily be shown that the time rate of change of $\boldsymbol{p}$, i.e. the velocity, is given by

$$\dot{\boldsymbol{p}} = \boldsymbol{\Omega} \times \boldsymbol{p}, \qquad (2.1)$$

where a superposed dot denotes a derivative with respect to time and $\boldsymbol{\Omega}$ is the angular velocity vector which is directed along the axis of rotation by the right-hand-rule and the magnitude of which is the speed of rotation. Consider now a moving frame defined, without loss of generality, by a Cartesian coordinate system $\{x, y, z\}$ with associated unit base vectors $\{\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}\}$, which is moving with respect to a fixed reference frame. Since the frame is moving, the set of unit base vector $\{\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}\}$ change with respect to time. Define a vector $\boldsymbol{\Omega}$ by

$$\boldsymbol{\Omega} = \Omega_x \boldsymbol{i} + \Omega_y \boldsymbol{j} + \Omega_z \boldsymbol{k},$$

$$\Omega_x = \frac{d\boldsymbol{j}}{dt} \cdot \boldsymbol{k} = -\frac{d\boldsymbol{k}}{dt} \cdot \boldsymbol{j}$$

$$\Omega_y = \frac{d\boldsymbol{i}}{dt} \cdot \boldsymbol{k} = -\frac{d\boldsymbol{k}}{dt} \cdot \boldsymbol{i}$$

$$\Omega_z = \frac{d\boldsymbol{i}}{dt} \cdot \boldsymbol{j} = -\frac{d\boldsymbol{j}}{dt} \cdot \boldsymbol{i} \qquad (2.2)$$

It follows from the definition (2.2) that

$$\frac{d\boldsymbol{i}}{dt} = \boldsymbol{\Omega} \times \boldsymbol{i};\ \frac{d\boldsymbol{j}}{dt} = \boldsymbol{\Omega} \times \boldsymbol{j};\ \frac{d\boldsymbol{k}}{dt} = \boldsymbol{\Omega} \times \boldsymbol{k}, \qquad (2.3)$$

Hence, the vector $\boldsymbol{\Omega}$ can be interpreted as the angular velocity vector directing along an axis about which the frame $\{\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}\}$ is rotating.

Next, consider a point in the frame $\{\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}\}$. The position vector of the point can be written as

$$\boldsymbol{r} = \boldsymbol{r}_0 + (\boldsymbol{r}_{/0})_{xyz}, \qquad (2.4)$$

where $\boldsymbol{r}_o$ is the position vector of the origin of the moving frame and $(\boldsymbol{r}_{/o})_{xyz}$ is the relative position vector as seen in the moving frame, i.e.

$$(\boldsymbol{r}_{/o})_{xyz} = x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}. \qquad (2.5)$$

Differentiating (2.5) yields

$$\boldsymbol{v} = \boldsymbol{v}_0 + (\boldsymbol{v}_{/0})_{xyz} + \boldsymbol{\Omega} \times (\boldsymbol{r}_{/0})_{xyz}, \qquad (2.6)$$

where $(\boldsymbol{v}_{/o})_{xyz}$ is the relative velocity seen within the moving frame, i.e.

$$(\boldsymbol{v}_{/0})_{xyz} = \dot{x}\boldsymbol{i} + \dot{y}\boldsymbol{j} + \dot{z}\boldsymbol{k} \qquad (2.7)$$

Differentiating (2.7) again yields the acceleration vector in the form

$$\boldsymbol{a} = \boldsymbol{a}_0 + (\boldsymbol{a}_{/0})_{xyz} + \dot{\boldsymbol{\Omega}} \times (\boldsymbol{r}_{/0})_{xyz} + 2\boldsymbol{\Omega}\ x\ (\boldsymbol{v}_{/0})_{xyz} + \boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times (\boldsymbol{r}_{/0})_{xyz}) \qquad (2.8)$$

where $\boldsymbol{a}_o$ is the acceleration of the origin of the frame, $\dot{\boldsymbol{\Omega}}$ is the rate of change of $\boldsymbol{\Omega}$ or the angular acceleration vector, and $(\boldsymbol{a}_{/o})_{xyz}$ is the relative acceleration as seen in the moving frame, i.e.

$$(\boldsymbol{a}_{/0})_{xyz} = \ddot{x}\boldsymbol{i} + \ddot{y}\boldsymbol{j} + \ddot{z}\boldsymbol{k} \qquad (2.9)$$

III. RIGID-BODY MOTION

Consider a rigid body in motion. Let a frame $\{\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}\}$ be attached to the body at point A in the body. The motion of the rigid body becomes the motion of the frame itself. Furthermore, since it has been shown that a moving frame has an associated angular velocity, the angular velocity of the frame $\{\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}\}$ becomes the angular velocity of the rigid body at the point A at which the frame is attached. Denote this angular velocity by $\boldsymbol{\omega}_A$. Because of the property of a rigid body, the relative position vector of any point of the rigid body with respect to point A remains unchanged regardless of the motion of the rigid body. Thus, within this attached frame the relative velocity and relative acceleration of any point as seen in the moving frame at A vanish, i.e.

$$(\boldsymbol{v}_{/0})_{xyz} = 0,$$

$$(\boldsymbol{a}_{/0})_{xyz} = 0, \qquad (3.1)$$

It follows from (2.6), (2.8), and (3.1) that the velocity and acceleration vectors of a point B of the rigid body are given by

$$\boldsymbol{v}_B = \boldsymbol{v}_A + \boldsymbol{\omega}_A \times \boldsymbol{r}_{B/A}$$

$$\boldsymbol{a}_B = \boldsymbol{a}_A + \dot{\boldsymbol{\omega}}_A\ x\boldsymbol{r}_{B/A} + \boldsymbol{\omega}_A \times (\boldsymbol{\omega}_A x\ \boldsymbol{r}_{B/A}) \qquad (3.2)$$

A similar derivation of equations (3.2) has been given in [5]. However, at present, it is possible that if another frame is attached to another point of the body, say C, the angular velocity of the new frame which will become the angular velocity of the body as seen from C may be different from $\boldsymbol{\omega}_A$ in (3.2). Let $\boldsymbol{\omega}_C$ be the angular velocity of the frame attached to the point C that becomes the angular velocity of the rigid body as seen from C. Similar to $(3.2)_1$, the velocity vector of the point B is given by

$$\boldsymbol{v}_B = \boldsymbol{v}_C + \boldsymbol{\omega}_C \times \boldsymbol{r}_{B/C}. \qquad (3.3)$$

Using $(3.2)_1$, the velocity $\boldsymbol{v}_C$ is given by

$$\boldsymbol{v}_C = \boldsymbol{v}_A + \boldsymbol{\omega}_A \times \boldsymbol{r}_{C/A}. \qquad (3.4)$$

Substituting (3.4) into (3.3) and equating the resulting expression with $(3.2)_1$, it follows that

$$(\boldsymbol{\omega}_C - \boldsymbol{\omega}_A)\ \mathrm{x} r_{B/C} = 0, \qquad (3.5)$$

where the basic relationship

$$r_{B/A} = r_{B/C} + r_{C/A},$$

has been used. Since B is any point in the body and since $\boldsymbol{\omega}_A$ and $\boldsymbol{\omega}_C$ do not depend on B, the necessary and sufficient condition for (3.5) to be true is that

$$(\boldsymbol{\omega}_A - \boldsymbol{\omega}_C) = 0, \quad (3.6)$$

or equivalently

$$\boldsymbol{\omega}_A = \boldsymbol{\omega}_C. \quad (3.7)$$

Since A and C are any two points in the rigid body, it follows that the choice of reference point where the frame is attached is immaterial and the angular velocity vector of the motion of a rigid body is unique and is the characteristic of the motion itself. Hence, the equations of rigid-body motion (3.2) reduce to (1.2) and (1.3), i.e.

$$\boldsymbol{v}_B = \boldsymbol{v}_A + \boldsymbol{\omega} \times \boldsymbol{r}_{B/A}.$$

$$\boldsymbol{a}_B = \boldsymbol{a}_A + \boldsymbol{\omega} \times \boldsymbol{r}_{B/A} + \boldsymbol{\omega}\ x\ (\boldsymbol{\omega} \times \boldsymbol{r}_{B/A}) \quad (3.8)$$

IV. CONCLUSIONS

It should be emphasized that the equations of rigid-body motion in the form (3.8) are a well-known result and can be proved in various ways. Here, the topic of motions in a moving frame is presented first to establish the existence of an angular velocity vector of the moving frame. By attaching a frame to a rigid body, the angular velocity of the frame becomes the angular velocity of the body at the point where the frame is attached. Utilizing the equations for motions in the attached frame, the expressions for velocity and acceleration of rigid-body motion can be easily obtained. The uniqueness of the angular velocity is then established and the general equations for rigid-body motion of the form (3.8) follow logically. All the derivation does not require any more mathematical knowledge than that in the first- or second-year undergraduate study and therefore is suitable for the instruction of undergraduate engineering mechanics.

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